QUATERNION-BASED SPARSE REPRESENTATION OF COLOR IMAGE SUPPLEMENTARY FILE

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1. QUATERNION ALGEBRA

Quaternion was firstly introduced by W. Hamilton [1] in 1843. It is of great interest because of its significance in vector analysis. Let $\dot{a} \in \mathbb{H}$ be a quaternion, then

$$\dot{a} = a_0 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k, \quad (1)$$

where $a_k \in \mathbb{R}, k = 0, 1, 2, 3$ are real numbers, and the imaginary units i, j, k obey the quaternion rules that $i^2 = j^2 = k^2 = -1$ and $i \cdot j = -j \cdot i = k, j \cdot k = -k \cdot j = i, k \cdot i = -i \cdot k = j$. As a vector entity, the quaternion is associative but non-commutative and its algebra can simultaneously manipulate all its four parts.

Let $\dot{a}$ and $\dot{b}$ be two quaternions, some quaternion algebras used in our paper are listed as follows,

1) Addition:

$$\dot{a}+\dot{b}=(a_0+b_0)+(a_1+b_1)\cdot i+(a_2+b_2)\cdot j+(a_3+b_3)\cdot k; \quad (2)$$

2) Multiplication:

$$\dot{a}\cdot\dot{b}=(a_0b_0-a_1b_1-a_2b_2-a_3b_3)+(a_0b_1+a_1b_0+a_2b_3-a_3b_2)\cdot i +(a_0b_2-a_1b_3+a_2b_0+a_3b_1)\cdot j+(a_0b_3+a_1b_2-a_2b_1+a_3b_0)\cdot k; \quad (3)$$

3) Norm:

$$\|\dot{a}\|=\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}\,; \quad (4)$$

4) Conjugate:

$$\overline{\dot{a}}=a_0-a_1\cdot i-a_2\cdot j-a_3\cdot k; \quad (5)$$

Another important algebra is the cross-correlation of two quaternion matrix. Given $\dot{\mathbf{I}}_1 \in \mathbb{H}^{M\times N}$ and $\dot{\mathbf{I}}_2 \in \mathbb{H}^{M\times N}$, the cross-correlation $C(m, n)$ is defined as follows,

$$C(m,n) = \sum_{p=0}^{M-1}\sum_{q=0}^{N-1} \dot{\mathbf{I}}_1(p,q)\cdot\overline{\dot{\mathbf{I}}_2(p,q)}, \quad (6)$$

where (p, q) is the row and column index of $\dot{\mathbf{I}}_1$ and $\dot{\mathbf{I}}_2$.

2. QUATERNION MATRIX SINGULAR VALUE DECOMPOSITION

Many applications of quaternion-based color image processing involves QSVD (quaternion matrix singular value decomposition) [2], which is also a key step in our algorithm.

Let $\dot{\mathbf{I}} \in \mathbb{H}^{M\times N}$ denotes a quaternion matrix, then $\dot{\mathbf{I}}$ can be represented by $\dot{\mathbf{I}} = A + B \cdot j$, where $A, B \in \mathbb{C}^{M\times N}$ are two complex matrices. Then we can convert $\dot{\mathbf{I}}$ into an equivalent complex matrix of $I_e \in \mathbb{C}^{2M\times 2N}$,

$$I_e = \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}_{2M\times 2N}, \quad (7)$$

In (7), the classical complex SVD algorithm can be directly applied to I_e. We denote these singular values as Λ', thus $I_e = U\Lambda' V^H$, where "H" is the Hermitian transpose operator, and U and V are two complex matrices.

In [2], the authors prove that for a quaternion matrix $\dot{\mathbf{I}}$ of rank r, there always exists two unitary quaternion matrices $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$, such that $\dot{\mathbf{I}} = \dot{\mathbf{U}}\Lambda_r\dot{\mathbf{V}}^H$, where $\Lambda_r = diag\{\lambda_1, \lambda_2, ..., \lambda_r\}$ and the λ's are the positive singular value of $I_{(q)}$. That is what we call QSVD of $\dot{\mathbf{I}}$.

The relation between the QSVD of a quaternion matrix and the SVD of its equivalent complex matrix is defined as follows,

$$\Lambda_r = row_{odd}(col_{odd}(\Lambda')),$$
$$\dot{\mathbf{U}} = col_{odd}(U^1) + col_{odd}(-\overline{U}^2)\cdot j,$$
$$\dot{\mathbf{V}} = col_{odd}(V^1) + col_{odd}(-\overline{V}^2)\cdot j,$$

where P^1 and P^2 denotes the upper half and lower half of matrix P, and $row_{odd}(P)$, $col_{odd}(P)$ means the odd rows and odd columns of matrix P respectively.

With quaternion description, more color information can be captured in the first several decompositions. Figure 1 shows the results of singular value distributions of SVD and QSVD for the same set of image block patches. As seen, the values obtained by QSVD decreases much faster than SVD for the first several points, indicating more information is contained in the first basis of $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$. As the first several decompositions are of more interest, this result, to some extent, implies QSVD is more adaptable to color signals than SVD. Thus when applied to K-QSVD, QSVD can make the dictionary update more efficient.

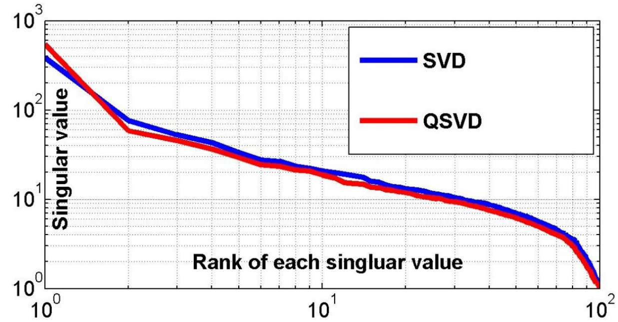

Fig. 1: Plots of ranked singular value distributions of SVD(blue) and QSVD(red). Both are implemented on 1000 color block patches of size 50×50. SVD is performed on the concatenation of RGB channels, and QSVD is on the quaternion expression of them.

3. REFERENCES

[1] William Rowan. Hamilton, "On quaternions, or on a new system of imaginaries in algebra," *Philosophical Magazine*, vol. 25, no. 3, pp. 489–495, 1844.

[2] J.H. Chang, J.J. Ding, et al., "Quaternion matrix singular value decomposition and its applications for color image processing," in *Image Processing, 2003. ICIP 2003. Proceedings. 2003 International Conference on*. IEEE, 2003, vol. 1, pp. I–805.